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On the representation of effective quantum field theories

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Abstract

We discuss a general model for effective quantum field theories (QFTs), which for example comprises quantum chromodynamics and quantum electrodynamics. We assume in the model a perturbative expansion of the Lagrangian with respect to a cut-off-energy parameter. For each order in the expansion we rigorously derive a representation, which is unitarily equivalent to the representation of the effective QFT in the non-interaction approximation. We interpret the various representations as different particle pictures with respect to which the effective QFT can be described. We further derive a rigorous scattering theory, which exactly reproduces results of formal scattering theory in quantum field theory.

Keywords: Effective Quantum Field Theory, Scattering Theory.

1 Introduction

Sensible QFTs are supposed to satisfy several mathematical requirements and physical principles. On the one hand, mathematical requirements are given by sets of axioms, most notably the Wightman axioms, the Osterwalder-Schrader axioms, and the Haag-Kastler axioms. Unfortunately only conventional free QFTs and some QFTs of interacting fields are known to obey these axioms so that many physically relevant QFTs are not confirmed from a strictly mathematical point of view.

On the other hand, physical principles such as Lorentz invariance and cluster decomposition impose a certain shape on QFTs [1, 2]. A remarkable fact is that physical principles naturally lead to interactions, which enforce a renormalization of the basic parameters of QFTs in order to ensure sensible results that agree with experiments. In particular, renormalization introduces an energy scale into a QFT on which it is supposed to be valid and for which the parameters are adjusted. QFTs should consequently be interpreted as effective theories rather than as fundamental theories [1, 2].

An effective QFT usually contains a cut-off-energy parameter that has a much larger value than typical energies on the energy scale in focus. If one writes down the most general Lagrangian for an effective QFT, which is allowed by the symmetries, then one basically obtains infinitely many interaction terms. Each term is however formally related to a power of the cut-off parameter, and grouping the terms with respect the order of the cut-off parameter one formally obtains a perturbative expansion. In particular, depending on up to which order, n ($n \in \mathbb{N}_0$), one considers one obtains an approximate Lagrangian, $\mathcal{L}_n$, containing finitely many interaction terms. However, each Lagrangian basically relates to the same effective QFT. So if we define with respect to each Lagrangian, $\mathcal{L}_n$, a representation of the effective QFT then it is consistent if these representations are unitarily related to each other. In particular, each representation would be unitarily equivalent to the representation corresponding to the non-interaction Lagrangian, $\mathcal{L}_0$, which actually defines a conventional free QFT.

In this paper, we explore representations of a general model QFT. In Sec. 2 we introduce the basic setting in which we define conventional free QFTs, and we illustrate the setting by the example of the non-interacting part of quantum chromodynamics. We use the representation of conventional free QFTs in the sequel to define representations of effective QFTs. To this end, we propose in Sec. 3 an interaction regularization by introducing appropriate cut-off parameters. We further use nonstandard analysis to implement a separation between energy scale in focus and cut-off energy. Nonstandard analysis turns out to be well suited for this purpose as it rigorously introduces infinitesimals and infinitely large numbers. Moreover, we work out the physical significance of the representations in the context of scattering theory in Sec. 5, and we finally summarize and discuss the results in Sec. 6.

2 Fock spaces of conventional free QFTs

2.1 General model

Definition 1: A particle system is a finite set S, on which a conjugation is defined,

$$\begin{array}{rcl} p & \to & \bar{p}, \qquad (p, \bar{p} \in S), \\ \bar{\bar{p}} & = & p. \end{array}$$

The elements of S are called particles and $\bar{p}$ is called the anti-particle of a particle $p \in S$.

We associate a conventional free quantum field theory with a particle system S as follows. Let us assume for each particle $p \in S$ a mass $m_p \in \mathbb{R}$ and a Hilbert space $\mathcal{H}_p$, which is unitarily equivalent to $L^2(H_{m_p}, \mu_m)$. We further assume that each particle $p \in S$ can either be classified as Boson or as Fermion. If p is a Fermion then we assume that the elements in $\mathcal{H}_p$ are classical solutions to the Dirac equation for particles of mass m_p. In particular, elements in $\mathcal{H}_p$ are spinors having spin $s = \pm 1/2$. If p is a massive Boson, $m_p \neq 0$, then we assume spin $s = 0$ or $s = \pm 1$, and unitary equivalence of $\mathcal{H}_p$ to $L^2(H_{m_p}, \mu_{m_p})$ basically means that the spinors also satisfy the Klein-Gordon equation in Minkowski spacetime. If p is a massless Boson, $m_p = 0$, then we do not associate spin with elements in $\mathcal{H}_p$. They rather are polarization vectors satisfying the wave equation $\Box\, v = 0$ in Minkowski spacetime.

However, let $\mathcal{F}_p$ denote either the fermionic Fock space or the Bosonic Fock space built upon $\mathcal{H}_p$. The total Fock space is defined by

$$\mathcal{F}_0 = \bigotimes_{p \in S} \mathcal{F}_p.$$

For a unitary operator, U_p, acting on $\mathcal{H}_p$ the unitary operator, $\Gamma_p(U_p)$, on $\mathcal{F}_p$ is defined by

$$(\Gamma_p(U_p)u_p)^{(n)} = \left(\bigotimes_{k=1}^{n} U_p \right) u_p^{(n)}, \quad (\Gamma_p(U_p)u_p)^{(0)} = u_p^{(0)},$$

for $u_p \in \mathcal{F}_p$.

Local raising and lowering 'operators' are defined as quadratic forms on the bosonic and fermionic Fock space over $L^2(\mathbb{R}^3)$, respectively. These quadratic forms can be promoted to operators by "smearing" with smooth functions, and for any unitary transform of $L^2(\mathbb{R}^3)$ we thus can define an

associated transform of the local raising and lowering 'operators'. In particular, $L^2(\mathbb{R}^3)$ is unitarily equivalent to each $L^2(H_{m_p}, \mu_{m_p})$ ($m_p \geq 0$), and the local raising and lowering 'operators' can further be transformed to quadratic forms on each $\mathcal{F}_p$ ($s \in I$) using the transformations $J_{m_p} : L^2(H_{m_p}, \mu_{m_p}) \to L^2(\mathbb{R}^3)$ given in Ref. [6]. We denote these quadratic forms in the following by $a_p^\dagger(\mathbf{p})$ and $a_p(\mathbf{p})$ ($\mathbf{p} \in \mathbb{R}^3$), respectively. Free local fields are further defined for $p = \bar{p}$ by

$$\Phi_{0,p}(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{\sqrt{\mu_p(\mathbf{p})}} \, e^{i(\mu_p(\mathbf{p})t - \mathbf{x}\mathbf{p})} a_p^\dagger(\mathbf{p}) + e^{-i(\mu_p(\mathbf{p})t - \mathbf{x}\mathbf{p})} a_p(\mathbf{p}),$$

and for $p \neq \bar{p}$ by

$$\Phi_{0,p}(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{\sqrt{\mu_p(\mathbf{p})}} \, e^{i(\mu_p(\mathbf{p})t - \mathbf{x}\mathbf{p})} a_{\bar{p}}^\dagger(\mathbf{p}) + e^{-i(\mu_p(\mathbf{p})t - \mathbf{x}\mathbf{p})} a_p(\mathbf{p}).$$

We note that $\mu_p(\mathbf{p}) = \sqrt{m_p^2 + \mathbf{p}^2}$ and that

$$\Phi_{0,\bar{p}}(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{d\mathbf{p}}{\sqrt{\mu_p(\mathbf{p})}} \, e^{i(\mu_p(\mathbf{p})t - \mathbf{x}\mathbf{p})} a_p^\dagger(\mathbf{p}) + e^{-i(\mu_p(\mathbf{p})t - \mathbf{x}\mathbf{p})} a_{\bar{p}}(\mathbf{p})$$

for $p \neq \bar{p}$. Furthermore, the total free Hamiltonian is given by

$$A_0 = \sum_{p \in S} A_{0,p} = \sum_{p \in S} \int d\mathbf{p} \, \mu_p(\mathbf{p}) a_p(\mathbf{p})^\dagger a_p(\mathbf{p}). \tag{1}$$

The above construction yields a representation, R_0, of a conventional free QFT satisfying the Wightman axioms [3]. As we will argue below, R_0 can actually be interpreted as a representation of an effective QFT in the non-interaction approximation.

2.2 Free quantum chromodynamics

In order to illustrate our general setting let us consider the example of quantum chromodynamics (QCD). The Lagrangian is given by

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F_{\mu\nu}^{(a)} F^{(a)\mu\nu} + i \sum_q \bar{\psi}_q^i ((D_\mu)_{ij} - m_q \delta_{ij}) \psi_q^j, \\
F_{\mu\nu}^{(a)} &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f_{abc} A_\mu^b A_\nu^c, \\
(D_\mu)_{ij} &= \delta_{ij} \partial_\mu + i g_s \sum_a \frac{\lambda_{ij}^a}{2} A_\mu^a,
\end{aligned}$$

where ψ_q^i are the 4-component Dirac spinors associated with each quark field of (3) color i and flavor q, and the A_μ^a are the (8) Yang-Mills (gluon) fields [4]. The Lagrangian can be split into a free part,

$$\begin{aligned} \mathcal{L}_0 &= -\frac{1}{4} F_{\mu\nu,0}^{(a)} F_0^{(a)\mu\nu} + i \sum_q \bar{\psi}_q^i (\partial_\mu - m_q) \psi_q^i, \\ F_{\mu\nu,0}^{(a)} &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a, \end{aligned}$$

and an interacting part, $\mathcal{L}_i = \mathcal{L} - \mathcal{L}_0$. Using the usual expansion with respect to local raising and lowering operators we can write the Yang-Mills fields as

$$\begin{aligned} A_\mu^a(\mathbf{x}) &= \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} \frac{d\mathbf{p}}{\sqrt{2|\mathbf{p}|}} \Big(e^{-i\mathbf{x}\mathbf{p}} (\tilde{a}_\mu^a)^\dagger(\mathbf{p}) + e^{i\mathbf{x}\mathbf{p}} \tilde{a}_\mu^a(\mathbf{p}) \Big), \\ \tilde{a}_\mu^a(\mathbf{p}) &= \sum_{j=1,2} \epsilon_{\mu,j}^a(\mathbf{p}) a_{\mu,j}^a(\mathbf{p}), \qquad \epsilon_{\mu,j}^a(\mathbf{p}) \epsilon_{\nu,i}^a(\mathbf{p}) = \delta_{ij}\delta_{\mu\nu}, \end{aligned}$$

where $\epsilon_\mu^a(\mathbf{p}, j)$ denote the usual polarization vectors. The single-particle Hilbert spaces associated with the indices μ, a are given by

$$\begin{aligned} \mathcal{H}_{\mu,a,j} &= \{ f_{\mu,a,j} = \epsilon_{\mu,j}^a f : \, f \in L^2(\mathbb{R}^3) \} \\ \langle f_{\mu,a,j}, g_{\mu,a,j} \rangle &= \langle f, g \rangle. \end{aligned}$$

Note that each $\mathcal{H}_{\mu,a,j}$ is unitarily equivalent to $L^2(\mathbb{R}^3)$ and to $L^2(H_0, d\Omega_0)$, and that the spaces $\mathcal{H}_{\mu,a,j}$ are orthogonal to each other. Analogously, we expand the Dirac spinors as

$$\begin{aligned} \psi_q^i(\mathbf{x}) &= \sum_{s=\pm} \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} d\mathbf{p} \\ & \sqrt{\frac{m_q}{E_q(\mathbf{p})}} \Big(e^{-i\mathbf{x}\mathbf{p}} u_q^{i,s}(\mathbf{p}) b_q^{i,s}(\mathbf{p}) + e^{i\mathbf{x}\mathbf{p}} v_q^{i,s}(\mathbf{p}) (d_q^{i,s}(\mathbf{p}))^\dagger \Big) \\ \bar{\psi}_q^{i,s}(\mathbf{x}) &= \sum_{s=\pm} \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} d\mathbf{p} \\ & \sqrt{\frac{m_q}{E_q(\mathbf{p})}} \Big(e^{i\mathbf{x}\mathbf{p}} \bar{u}_q^{i,s}(\mathbf{p}) (b_q^{i,s}(\mathbf{p}))^\dagger + e^{-i\mathbf{x}\mathbf{p}} \bar{v}_q^{i,s}(\mathbf{p}) d_q^{i,s}(\mathbf{p}) \Big), \\ E_q(\mathbf{p}) &= \sqrt{m_q^2 + \mathbf{p}^2}, \\ \bar{u}_q^{i,s}(\mathbf{p}) u_{q'}^{j,t}(\mathbf{p}) &= -\bar{v}_q^{i,s}(\mathbf{p}) v_{q'}^{j,t}(\mathbf{p}) = \delta_{ij}\delta_{qq'}\delta_{st} \\ \bar{u}_q^{i,s}(\mathbf{p}) v_{q'}^{j,t}(\mathbf{p}) &= \bar{v}_q^{i,s}(\mathbf{p}) u_{q'}^{j,t}(\mathbf{p}) = 0. \end{aligned}$$

The single-particle Hilbert spaces associated with the indices q, i, s are given by

$$\mathcal{H}_{q,i,s} = \{ f_{q,i,s} = u_q^{i,s} f : \, f \in L^2(\mathbb{R}^3) \}$$

$$\begin{aligned}\mathcal{H}'_{q,i,s} &= \{f'_{q,i,s} = v_q^{i,s} f : f \in L^2(\mathrm{I\!R}^3)\} \\ \langle f_{q,i,s}, g_{q,i,s}\rangle &= \langle f'_{q,i,s}, g'_{q,i,s}\rangle = \langle f, g\rangle.\end{aligned}$$

Note that each $\mathcal{H}_{q,i,s}$ and each $\mathcal{H}'_{q,i,s}$ is unitarily equivalent to $L^2(\mathrm{I\!R}^3)$ and to $L^2(H_{m_q}, d\Omega_{m_q})$, and that the spaces $\mathcal{H}_{q,i,s}, \mathcal{H}'_{q,i,s}$ are orthogonal to each other. Moreover, the free Hamiltonian is given by

$$\begin{aligned}A_0 &= \sum_{\mu,a} \int d\mathbf{p}\, |\mathbf{p}| a^a_{\mu,j}(\mathbf{p})^\dagger a^a_{\mu,j}(\mathbf{p}) + \\ &\quad \sum_{q,i,s} \int d\mathbf{p}\, E_q(\mathbf{p}) \Big(b_q^{i,s}(\mathbf{p})^\dagger b_q^{i,s}(\mathbf{p}) + d_q^{i,s}(\mathbf{p})^\dagger d_q^{i,s}(\mathbf{p}) \Big).\end{aligned}$$

3 Interaction regularization

Classical Hamiltonians normally cannot directly be promoted to operators on a Hilbert space in QFTs. The first step is therefore to introduce a Wick ordering of raising and lowering operators so that the resulting expressions are defined as quadratic forms. However, Wick-ordered expressions for Hamiltonians are usually still too singular to allow a definition as operators on a Hilbert space. For this reason, one usually further approximates the singular expressions by "cutting them off" or "butchering" [3], and the proper operator definition of the Hamiltonian is then achieved by a limiting procedure. We proceed in this paper in an analogous manner.

Let $\{h_n\}_{n\in\mathrm{I\!N}}$ be the basis of Hermite functions of $L^2(\mathrm{I\!R}^3$ [5]. As each Fock space $\mathcal{F}_p$ is built upon a single-particle Hilbert space $\mathcal{H}_p$, which is unitarily equivalent to $L^2(\mathrm{I\!R}^3)$, we can use the Hermite-function basis, $\{h_n\}_{n\in\mathrm{I\!N}}$, to define a basis $\{u_n\}_{n\in\mathrm{I\!N}}$ on $\mathcal{F}_0$. We assume that the basis $\{u_n\}_{n\in\mathrm{I\!N}}$ has sufficiently "nice" properties in the following sense: As we have pointed out in Sec. 2 for the example of QCD, a formal Hamiltonian expression, A, can usually be divided into a free part, A_0, and an interacting part, A_i. A_0 is a well-defined operator on $\mathcal{F}_0$ [3, 6] given by Eq. (1) whereas A_i is only defined as a quadratic form, which we denote by $A_i(\cdot,\cdot)$ in the sequel. We assume that A can be approximated in the sense of quadratic forms by the operators

$$A_n = A_0 + \sum_{j,k\leq n} A_i(u_j, u_k)\; |u_j\rangle\langle u_k|.$$

In particular, we assume that $A_i(u_j, u_k)$ is well defined for $j, k \in \mathrm{I\!N}$, which is one of the "nice" properties of the basis $\{u_n\}_{n\in\mathrm{I\!N}}$. We will however come back to this point later.

However, regularizations are used in QFT to tame singular integral expressions. Let us also assume a renormalization scheme that introduces a regulator r and that can also be applied to the quadratic form $A_i(\cdot,\cdot)$, which is

basically defined by integral expressions. We denote the regularized quadratic form by $A_{i,r}(\cdot,\cdot)$ and obtain in this way the regularized operators

$$A_{n,r} = A_0 + \sum_{j,k\leq n} A_{i,r}(u_j, u_k)\ |u_j\rangle\langle u_k|.$$

We assume further that a renormalization scheme is applied. This usually implies that one introduces physical masses that are adjusted with respect to a specific energy scale. The masses in the free Hamiltonian A_0 are replaced (if necessary) by the physical masses, and potential counter terms are included in A_i [1]. This step is necessary to ensure a sensible physical spectrum for $A_{n,r}$ and A_0.

In order to illustrate the setting further let us come back to the example of QCD. Let $\mathcal{L}_r$ denote the transferred renormalized Lagrangian. Note that the structure of the Lagrangian does not change and that we can write it as a sum of a free part, $\mathcal{L}_0$, and an interaction part, $\mathcal{L}_{i,r}$, analogously as in Sec. 2. Note also that $\mathcal{L}_0$ now contains physical masses, which are adjusted to a specific energy scale, and that the counter terms are included in $\mathcal{L}_{i,r}$. Appropriate equations for counter terms are given in [7]. Moreover, the interaction Hamiltonian expression is given in this example by $A_{i,r} = -:\mathcal{L}_{i,r}:$ where normal ordering is indicated as usual by colons.

The main advantage of introducing the approximate Hamiltonians $A_{n,r}$ is that scattering theory can rigorously be applied. Let $\mathcal{F}_{ac}$ denote the subspace of $\mathcal{F}_0$ that is related to the absolutely continuous spectrum of A_0, let P_{ac} denote the projection onto $\mathcal{F}_{ac}$, and let w_0 denote the ground state of the free theory,

$$w_0 = \bigotimes_{p\in S} w_{0,p} \qquad (\Phi_{0,p}(\mathbf{x},t)w_{0,p} = 0 \quad \forall p \in S).$$

We note that w_0 is the only eigenvector of A_0, and that the spectrum of $A_0|_{\{w_0\}^\perp}$ is absolutely continuous, $\{w_0\}^\perp = \mathcal{F}_{ac}$. Moreover, since the operator $(A_{n,r} - A_0)$ has hyperfinite rank, the Møller wave operators,

$$W_{\pm,n,r} = \text{s-}\lim_{t\to\mp\infty} e^{iA_{n,r}t}e^{-iA_0t}\,P_{ac},$$

exist by the Kato-Rosenblum theorem and the quantum system is complete [8]. We further extend the operators $W_{\pm,n,r}$ by defining $W_{\pm,n,r}\,w_0 = w_0$, which is consistent due to the Wick-ordering of the interaction. Note that the operators $W_{\pm,n,r}$ are partial isometries. The ranges of the operators $W_{\pm,n,r}$, $\mathcal{F}_{\pm,n,r} = W_{\pm,n,r}(\mathcal{F}_0)$, coincide since the quantum system is complete. Let $P_{n,r}$ be the corresponding projection of $\mathcal{F}_0$ onto $\mathcal{F}_{\pm,n,r}$. We extend the

adjoint operators $W^\dagger_{\pm,n,r}$ to operators on $\mathcal{F}_0$ by defining $W^\dagger_{\pm,n,r} = W^\dagger_{\pm,n,r} P_{n,r}$, which is consistent since

$$\langle u, W^\dagger_{\pm,n,r} P_{n,r} v\rangle = \langle P_{n,r} W_{\pm,n,r} u, v\rangle = \langle W_{\pm,n,r} u, v\rangle$$

for $u, v \in \mathcal{F}_0$. Moreover,

$$A_{n,r}|_{D(A_{n,r})\cap \mathcal{F}_{\pm,n,r}} = W_{\pm,n,r} A_0 (W_{\pm,n,r})^\dagger$$

and the spaces $\mathcal{F}_{\pm,n,r}$ are invariant with respect to $A_{n,r}$.

4 Approaching the full interaction

In order to calculate quantitative results in renormalized QFTs one considers limits with respect to the regulator. From a physical perspective this normally means that one uses an energy cut-off at a much larger energy than given by the energy scale in focus. For the sake of simplicity let us assume that the limit is given by $r \to \infty$. As mentioned in Sec. 1, we can conveniently use nonstandard analysis to consider the limit and to achieve a separation of cut-off energy and target energy scale. One of the main advantages of nonstandard analysis is that it provides a framework, in which infinitesimals and infinitely large numbers are rigorously defined. In particular, we say that an energy, E_1, is much larger than another energy, E_2, if E_1/E_2 is infinitely large. Equivalently we can write $E_1 \gg E_2$ or $E_2/E_1 \approx 0$. The occurrence of infinitesimals or infinitely large numbers in a nonstandard framework usually corresponds to a limiting process in a standard framework, and vice versa. For example, considering the limit $r \to \infty$ in a standard framework is equivalent to choosing infinite hyperreals, r, in a nonstandard framework. Note that the set of hyperreals in a nonstandard framework corresponds to reals in a standard framework.

We assume in the sequel a polysaturated extension of a superstructure, $\mathcal{V}$, that contains all standard mathematical objects of interest [9]. In particular, $\mathcal{V}$ contains real numbers, complex numbers, functions, etc., and the (nonstandard) extension of $\mathcal{V}$ is given by an injective mapping of $\mathcal{V}$ onto another superstructure $\mathcal{W}$, $^* : \mathcal{V} \to \mathcal{W}$. We obtain by the extension, for example, the sets of hyperreal numbers, $^*\mathbb{R} \in \mathcal{W}$, and hypernatural numbers, $^*\mathbb{N} \in \mathcal{W}$, which can be used in the same manner as real numbers and natural numbers.

In a nonstandard setting, the transferred renormalized Hamiltonian is given by

$$A_{n,r} = {}^*A_0 + \sum_{j,k\le n} {}^*A_{i,r}(u_j, u_k)\ |u_j\rangle\langle u_k| \quad (n \in {}^*\mathbb{N}).$$

Note that ${}^*A_0$ is the transferred Hamiltonian of conventional free fields, which acts on the internal Hilbert space ${}^*\mathcal{F}_0$.

As mentioned in Sec. 3, the full Hamiltonian in standard QFT is formally only given as a quadratic form, A_r, for finite values of the regulator, r. Calculable quantities in QFT are basically functions of r, which are defined by integrals. For example, for vectors u, v in the domain of $A_r(\cdot,\cdot)$ we can define the integral (function) $g(r) = A_r(u,v)$. Such integrals (functions) can further be written as $g(r) = \lim_n g_n(r)$ with $g_n(r) = A_{n,r}(u,v)$, i.e., we use the regularization introduced in Sec. 3. The limit is well-defined because of the "nice" properties of the basis $\{u_n\}_{n\in\mathbb{N}}$, which is defined with the help of Hermite functions.

Let $\mathcal{G}$ denote the set of functions with the property that $\bar{g} = \lim_r g(r)$ exists and that $\bar{g} = \lim_r \lim_n g_n(r)$, where $g_n(r)$ is a function of r with respect to the truncated basis $\{u_m\}_{m\leq n}$. We have good reason to assume that $\mathcal{G}$ contains all functions of interest, c.f. Sec. 5. However, we treat these limits now by nonstandard analysis, assuming that r has a fixed infinite value, i.e., ${}^*\bar{g} \approx {}^*g(r)$ for all $g \in \mathcal{G}$.

Theorem 1: There exists a hypernatural number $h \in {}^*\mathbb{N}$ so that ${}^*\bar{g} \approx {}^*g_h(r)$, i.e., $\bar{g} = {}^o({}^*g_h(r))$, holds for all $g \in \mathcal{G}$.

Proof: The statement ${}^*g(r) = {}^*\lim_n {}^*g_n(r)$ is equivalent to

$$(\forall \epsilon > 0)(\exists n_{\epsilon,g} \in {}^*\mathbb{N})(\forall m \geq n_{\epsilon,g}) \quad |{}^*g(r) - {}^*g_m(r)| < \epsilon.$$

Let $\epsilon > 0$ be infinitesimal, i.e., ${}^o\epsilon = 0$, and let $L_{\epsilon,g} = \{m \in {}^*\mathbb{N} : m \geq n_{\epsilon,g}\}$. The sets $L_{\epsilon,g}$ are internal, and by polysaturation there exists a hypernatural number with the property

$$h \in \bigcap_{g\in\mathcal{G}} L_{\epsilon,{}^*g},$$

i.e., $(\forall g \in \mathcal{G}) \quad |{}^*g(r) - {}^*g_h(r)| < \epsilon$, and ${}^*\bar{g} \approx {}^*g_h(r)$ thus holds for all $g \in \mathcal{G}$.
■

Theorem 1 guarantees that the operator ${}^*A_{h,r}$ reliably models the full interaction. We note that the subscripts h and r indicate that we are dealing with an effective theory. h and r are basically related to a cut-off energy, and as h and r are both infinitely large we are effectively working at an energy scale far below the cut-off energy scale.

Transferring the results of scattering theory in Sec. 3 we see that the restriction of ${}^*A_{h,r}$ to the space ${}^*\mathcal{F}_{\pm,h,r} = {}^*W_{\pm,h,r}({}^*\mathcal{F}_0)$ is unitarily equivalent to the Hamiltonian ${}^*A_0$,

$${}^*A_{h,r}|_{D({}^*A_{h,r})\cap {}^*\mathcal{F}_{\pm,h,r}} = {}^*W_{\pm,h,r}\, {}^*A_0\, {}^*(W_{\pm,h,r})^\dagger\,. \tag{2}$$

We can therefore use the operators ${}^*W_{\pm,h,r}$ to define a representation of the effective QFT associated with the Lagrangian $\mathcal{L}_r = \mathcal{L}_0 + \mathcal{L}_{i,r}$.

The free Hamiltonian, A_0, which we obtain if we omit the interaction in the Lagrangian and consider physical masses in the non-interaction part, acts on the Fock space $\mathcal{F}_0$. As in Sec. 2 we denote the associated representation of the conventional free QFT by R_0. Moreover, the operator ${}^*A_{h,r}$ is defined as an approximation on the nonstandard Fock space ${}^*\mathcal{F}_0$. The approximation vanishes however once we take standard parts or nonstandard hulls. The rationale for introducing nonstandard hulls is that measurements can be carried out with at most standard precision. An outline of nonstandard hulls is given in the appendix.

Let $W = {}^o({}^*W_{+,h,r})$, then W is a unitary operator acting on ${}^o({}^*\mathcal{F}_0)$. As $\mathcal{F}_0 \subset {}^o({}^*\mathcal{F}_0)$, the operator W induces a representation R_i, which is unitarily equivalent to R_0. In particular, the Hamiltonian $A = WA_0W^\dagger$ is a restriction of ${}^o({}^*A_{h,r})$,

$$A = {}^o({}^*W_{+,h,r})\,{}^o({}^*A_0){}^o({}^*W_{+,h,r})^\dagger|_{D(A)=W(D(A_0))} = {}^o({}^*A_{h,r})|_{D(A)}$$

We will discuss the physical interpretation of representations R_0 and R_i in Sec. 6.

However, the representations R_0 and R_i are defined on Fock spaces $\mathcal{F}_0$ and $W(\mathcal{F}_0)$, which are subspaces of the Fock space ${}^o({}^*\mathcal{F}_0)$. Since the Hamiltonians A_0 and A are restrictions of ${}^o({}^*A_0)$ and ${}^o({}^*A_{h,r})$, respectively, we obtain

$$\begin{aligned} \exp(iA_0t) &= {}^o\exp(i\,{}^*A_0\,{}^*t)|_{\mathcal{F}_0} = {}^o({}^*U_0({}^*t))|_{\mathcal{F}_0} \\ \exp(iAt) &= {}^o\exp(i\,{}^*A_{h,r}\,{}^*t)|_{W(\mathcal{F}_0)} = {}^o({}^*U_{h,r}({}^*t))|_{W(\mathcal{F}_0)}\,. \end{aligned}$$

We gain two advantages by considering the extensions ${}^o({}^*U_0(\cdot))$ and ${}^o(U_{h,r}(\cdot))$. First, both operators are defined on the same Hilbert space ${}^o({}^*\mathcal{F}_0)$. We can therefore consider the dynamics of the two representations in one Hilbert space. Second, the operators refer to nonstandard arguments and we can therefore conveniently consider different time scales as mentioned in Sec. 1. Both advantages will be used in Sec. 5 when we establish a nonstandard scattering theory.

5 Scattering theory

Scattering theory describes processes on a spacetime scale, which is large compared to the spacetime scale of the actual interaction. As mentioned in Sec. 1, different physical scales can conveniently be modeled by nonstandard analysis. Transferring the results of Sec. 3, the nonstandard scattering

operator is given by

$$\begin{aligned} {}^*S_{h,r} &= {}^*W_{-,h,r}^{\dagger}\, {}^*W_{+,h,r} \\ {}^*W_{\pm,h,r} &= \text{s-}\lim_{t\to\mp{}^*\infty} \exp(i\,{}^*A_{h,r}t)\exp(-i\,{}^*A_0 t)\,{}^*P_{ac}. \end{aligned}$$

Let $S = {}^o({}^*S_{h,r}) = {}^o({}^*W_{-,h,r})^{\dagger}\, {}^o({}^*W_{+,h,r})$.

Theorem 2: There exists a $T > 0$ so that for all $u \in \mathcal{F}_0$,

$${}^o({}^*W_{+,h,r})u = {}^o({}^*U_{h,r}(t))^{\dagger}\, {}^o({}^*U_0(t))u$$

holds for all $t < -T$, and that

$${}^o({}^*W_{-,h,r})u = {}^o({}^*U_{h,r}(t))^{\dagger}\, {}^o({}^*U_0(t))u$$

holds for all $t > T$.

Proof: Let $\epsilon > 0$ be infinitesimal. For each $u \in \mathcal{F}_0$ there exists a $T_u > 0$ for which

$$\|[{}^*W_{+,h,r} -{}^* U_{h,r}(t)^{\dagger}\, {}^*U_0(t)]\, {}^*u\| < \epsilon$$

holds for $t < -T_u$. Since each interval $[T_u, {}^*\infty)$ is internal there exists a T_+ in

$$\bigcap_{u\in\mathcal{F}_0} [T_u, {}^*\infty)$$

by polysaturation, and for $t < -T_+$ we obtain

$$\|[{}^*W_{+,h,r} -{}^* U_{h,r}(t)^{\dagger}\, {}^*U_0(t)]\, {}^*u\| \approx 0.$$

Analogously, there exists a T_- so that

$$\|[{}^*W_{-,h,r} -{}^* U_{h,r}(t)^{\dagger}\, {}^*U_0(t)]\, {}^*u\| \approx 0$$

holds for all $u \in \mathcal{F}_0$ and $t > T_-$. Choosing $T = \max\{T_-, T_+\}$ we obtain the assertion. ∎

Theorem 2 allows us to introduce explicit starting and ending times for scattering processes. In a scattering experiment, the preparation and detection take place in the remote past and far future, which are modeled in a standard theory by the limits $t \to \pm\infty$. Motivated by theorem 2, we can quantify the terms 'remote past' and 'far future', i.e., the preparation and detection take place at times $\mp T$. The set of physical preparation states is then contained in ${}^o({}^*U_0(-T))(\mathcal{F}_0)$ and the set of physical detection states in ${}^o({}^*U_0(T))(\mathcal{F}_0)$.

The transition probability from a state $u \in \mathcal{F}_0$ to a state $v \in \mathcal{F}_0$ is $|\langle u, Sv \rangle|^2$, where S is the scattering operator. More precisely, $|\langle u, Sv \rangle|^2$ is the probability that a state, which looked like the free state u in the remote past (i.e., at time $-T$) will look like the free state v in the far future (i.e., at time T). The scattering operator in standard QFTs is usually defined only as a formal expression, and we discuss now how such formal expressions are related to the scattering operator defined before theorem 2. Let

$$U_{h,r}(t,t') = \exp(i\,{}^*A_0 t)\, \exp(-i\,{}^*A_{h,r} t)\, \exp(i\,{}^*A_{h,r} t')\, \exp(-i\,{}^*A_0 t')$$

and let

$$A_{i,h,r}(t) = \exp(i\,{}^*A_0 t)\, ({}^*A_{h,r} - {}^*A_0)\, \exp(-i\,{}^*A_0 t).$$

$A_{i,h,r}(t)$ is a hyperfinite-rank operator and

$$\frac{d}{dt} U_{h,r}(t,t') = -iA_{i,h,r}(t) U_{h,r}(t,t')$$

is defined on ${}^*\mathcal{F}_0$. We can expand $U_{h,r}(t,t')$ in a Dyson series for $t \geq t'$,

$$\begin{aligned}
U_{h,r}(t,t') &= 1 - i \int_{t'}^{t} dt_1 A_{i,h,r}(t_1) U_{h,r}(t_1,t') \\
&= 1 + \sum_{n=1}^{{}^*\infty} (-i)^n \int_{t'}^{t} dt_1 ... \int_{t'}^{t_{n-1}} dt_n A_{i,h,r}(t_1) \cdot ... \cdot A_{i,h,r}(t_n).
\end{aligned}$$

Using the common time-ordering operator [7, 10], $\mathcal{T}$, we obtain

$$\begin{aligned}
U_{h,r}(t,t') &= 1 + \sum_{n=1}^{{}^*\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} dt_1 ... \int_{t'}^{t} dt_n \mathcal{T}(A_{i,h,r}(t_1) \cdot ... \cdot A_{i,h,r}(t_n)) \\
&= \mathcal{T} \exp\left(-i \int_{t'}^{t} dt'' A_{i,h,r}(t'')\right),
\end{aligned}$$

and

$$\begin{aligned}
{}^*S_{h,r} &= \text{s-}\lim_{t \to {}^*\infty} {}^*\text{s-}\lim_{t' \to -{}^*\infty} U_{h,r}(t,t') \qquad (3) \\
&= \mathcal{T} \exp\left(-i \int_{-{}^*\infty}^{{}^*\infty} dt'' A_{i,r}(t'')\right),
\end{aligned}$$

which is an analogous expression for the scattering operator as known from standard QFT [1, 7, 10].

In particular, let $u, v \in \mathcal{F}_0$ and let us assume a formal standard scattering operator, S_r, which depends on the regulator r. The matrix element $\langle u, S_r v\rangle$ defines a function $g(r)$. Let us assume that in the standard theory $\bar{g} = \lim_r g(r)$ exists and that $\bar{g} = \lim_r \lim_n g_n(r)$ with the basis $\{u_n\}_{n\in\mathbb{N}}$ appropriately inserted, then ${}^*\bar{g} \approx {}^*g(h, r)$ holds for the fixed infinite values of r and h assumed in this paper by theorem 1. Since ${}^*g(h, r) = \langle {}^*u, {}^*S_{h,r}\, {}^*v\rangle$ we obtain $\bar{g} = \langle u, Sv\rangle$, i.e., the scattering operator, S, which is rigorously defined in our scattering theory, has the same matrix elements as the formal scattering operator in standard QFT. We note that we obtain an analogous result if we consider only a finite number of terms in Eq. (3) and compare them to formal standard expressions.

6 Discussion

We discuss in this paper the representation of effective QFTs. Effective QFTs are described by the most general Lagrangian, $\mathcal{L}$, which is consistent with the symmetries of the theory, so that $\mathcal{L}$ basically contains infinitely many interaction terms [1, 2]. An effective QFT is further assumed to be valid only on a specific energy scale, and one typically introduces a cut-off-energy parameter into the theory, which has a much higher value as compared to energies on the energy scale in focus. The interaction terms in $\mathcal{L}$ can be formally grouped with respect to the order of the cut-off parameter, and one formally obtains a perturbative expansion of $\mathcal{L}$. Depending on up to which order, n ($n \in \mathbb{N}_0$), one considers in the expansion of $\mathcal{L}$ one obtains in this way approximate Lagrangians, $\mathcal{L}_n$. In particular, the non-interaction approximation Lagrangian, $\mathcal{L}_0$, describes a conventional free QFT.

However, as the approximate Lagrangians, $\mathcal{L}_n$, all relate to the same effective QFT, we believe that it is consistent if we define for each $\mathcal{L}_n$ a representation R_n so that the representations are unitarily equivalent to each other, $R_n \sim R_m$ ($n, m \in \mathbb{N}_0$). We work out this idea by introducing in Sec. 2 a general free QFT, which comprises QED and QCD in the non-interaction approximation as special cases. The representation of free QFTs is well-known [3] and serves as a starting point of our construction. In Sec. 3 we further consider a general interaction Lagrangian, which could be one of the approximate Lagraginas emerging from an effective QFT approach. We introduce certain regularizations, i.e., cut-off parameters, and demonstrate how conventional scattering theory can be used to establish a rigorous representation, R_i, which is unitarily equivalent to the non-interaction representation, R_0.

In order to obtain a proper separation of energy scales, we apply nonstan-

dard analysis and consider infinite values of the cut-off parameter in Sec. 4. As mentioned earlier, nonstandard analysis is well-suited for this purpose since it rigorously introduces infinitesimals and infintely large numbers. For example, an energy E_1 is on a larger energy scale than an energy E_2, if E_2/E_1 is infinitesimal. We note that the occurrence of infinitesimals and infinitely large numbers in a nonstandard framework usually corresponds to a limiting procedure in a standard framework. In our example, we would start with finite energies E_1 and E_2, and would consider the limit $E_1 \to \infty$.

However, we define in Sec. 4 the representation corresponding to the general interaction Lagrangian by considering appropriate infinite values of the cut-off parameters. We further show in Sec. 5 that the representation is sensible with respect to (nonstandard) scattering theory. We note that we derive nonstandard scattering theory in this paper in an analogous manner as for quantum mechanical systems in Ref. [11]. The motivation for using nonstandard analysis for scattering theory is again a separation of scales: Scattering processes are observed on a spacetime scale, which is large compared to the spacetime scale of the actual interaction. In our scattering theory, we can explicitly identify the starting time, $-T$, and the ending time, T, of the scattering process. The time T is typically infinite and refers to a different time scale than the time scale of the standard QFT representations. More important is however the fact that the scattering operator has exactly the same matrix elements as the formal scattering operator in standard QFT. For this reason, we believe that the representation of the effective QFT is not only mathematically rigorous but also physically valid.

Let us finally discuss how the representations, R_n, emerging from a formal perturbative expansion of the Lagrangian of an effective QFT can be interpreted, especially as they are all unitarily equivalent to the non-interaction representation, R_0. An important insight by the Unruh effect is that different representations of a conventional free QFT basically express different notions of what the free particles of the theory are [12]. In quantum electrodynamics, for example, the free particles at low energies (approximately) are the electron and the photon, but at higher energies these particles can never be observed alone, and other electrons or photons are always in their neighbourhood. We therefore cannot consider electrons and photons as free particles at higher energies, since we rather observe composite states of these particles. Another example is quantum chromodynamics, which exhibits the feature of asymptotic freedom. Quarks can (approximately) be seen as free particles at very large energies. For lower energies, however, their coupling becomes stronger and we observe only composite quark states. So in general, the representation R_0 of an effective QFT is rather related to a benchmark particle picture, which is used to describe particle states in the other repre-

sentations, and especially single-particle states in the other representations occur as compounds of single particles in R_0.

Appendix

We summarize in the following a few basic facts about nonstandard hulls as they can be found in Refs. [9, 11], for example.

Nonstandard hulls

Let $\mathcal{H}$ be a nonstandard inner-product space. The definition of the nonstandard hull ${}^o\mathcal{H}$ of the space $\mathcal{H}$ introduces an equivalence relation on the set of finite nonstandard vectors, $\mathrm{fin}(\mathcal{H}) = \{x \in \mathcal{H} : \|x\| \in \mathrm{fin}({}^*\mathbb{R})\}$. We note that $\mathrm{fin}({}^*\mathbb{R})$ is the set of finite hyperreals. Two vectors are equivalent if their difference has infinitesimal norm. The nonstandard hull ${}^o\mathcal{H}$ of the space $\mathcal{H}$ is given as the quotient

$$\begin{aligned} {}^o\mathcal{H} &= \mathrm{fin}(\mathcal{H})/\mathcal{H}_0, \quad \mathcal{H}_0 = \{x \in \mathcal{H} : \|x\| \approx 0\}, \\ \|{}^o x\| &= \mathrm{st}(\|x\|), \quad \langle {}^o x, {}^o y\rangle = \mathrm{st}(\langle x, y\rangle). \end{aligned}$$

In particular, ${}^o\mathcal{H}$ is a Hilbert space. An important fact is that for a standard Hilbert space $\mathcal{H}$ with nonstandard extension ${}^*\mathcal{H}$ the nonstandard hull, ${}^o({}^*\mathcal{H})$, contains $\mathcal{H}$ as subspace, ${}^o({}^*\mathcal{H}) \supseteq \mathcal{H}$. We note that $\mathcal{H}$ is a proper subset of ${}^o({}^*\mathcal{H})$ if $\mathcal{H}$ is infinite dimensional.

Furthermore, nonstandard hulls are also defined for internal finitely-bounded linear operators. Let A be such an operator on $\mathcal{H}$, then its nonstandard hull is defined as

$$\begin{aligned} {}^oA\,{}^o x &= {}^o(Ax) \quad \forall\, x \in \mathrm{fin}(\mathcal{H}) \\ \|{}^oA\| &= {}^o\|A\|. \end{aligned}$$

In particular, if A is self-adjoint then ${}^oA$ is also self-adjoint.

Moreover, in Ref. [11] the definition of nonstandard hulls is extended to self-adjoint internal nonstandard operators, which are not necessarily finitely bounded. The definition is based on Loeb-function calculus and is obtained as follows. Let A be a self-adjoint internal nonstandard operator on an internal Hilbert space, then the transferred spectral theorem yields a projection-valued *measure, $E_\Omega\,(\Omega \in {}^*\mathcal{B})$, on the transferred Borel subsets of $\mathbb{R}$, ${}^*\mathcal{B}$, and

$$A = \int_{{}^*\mathbb{R}} x\, dE\,.$$

The transferred Borel sets, ${}^*\mathcal{B}$, which form a ${}^*\sigma$-algebra, are contained in the σ-algebra of universally Loeb-measurable sets, $\mathcal{A}$, and the projection-valued *measure, E_Ω ($\Omega \in {}^*\mathcal{B}$), extends to a projection-valued Loeb-measure, ${}^oE_\Omega$ ($\Omega \in \mathcal{A}$). In particular, ${}^oE_\Omega$ is the usual nonstandard hull of E_Ω for $\Omega \in {}^*\mathcal{B}$. The standard-part function, $\mathrm{st}(\cdot)$, is Loeb-measurable, i.e., $\mathrm{st}^{-1}(\mathcal{B}) \subset \mathcal{A}$, and the nonstandard hull of A is defined as

$$ {}^oA = \int_{\mathrm{fin}({}^*\mathbb{R})} \mathrm{st}\, d\,{}^oE. $$

We note that if A is finitely bounded then ${}^oA$ is just the usual nonstandard hull. Moreover, since ${}^oA$ is self-adjoint it also has a spectral representation,

$$ {}^oA = \int_{\mathbb{R}} \omega\, dF_\omega, $$

which is given by the standard spectral theorem. The relationship of the projection-valued measures dF and $d\,{}^oE$ is given by $F(\Omega) = {}^oE(\mathrm{st}^{-1}(\Omega))$ for Borel sets $\Omega \in \mathcal{B}$.

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